Name of college — S.S. College, J-Bad (1)
Deptt — Mathematics
Topic — Problem based on Gauss's Test (Infinite Series)
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Problem based on Gauss Test —

Since this Test is applicable when Raabe's Test fails or even directly and thus avoiding Ratio and Raabe's Tests which are special case of Gauss Test.

Meaning of $O(\frac{1}{n^2})$ → $O(\frac{1}{n^2})$ means the term containing $1/n^2$ and powers of $1/n$ higher than $\frac{1}{n^2}$

Ex:→ Test the series :→

$$\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\cdots$$

Solution → Given series is

$$\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\cdots$$

Let u_n be the nth Term of the series

then $u_n = \dfrac{1^2\cdot 3^2\cdot 5^2 \cdots (2n-1)^2 \, (\cancel{\;})^2}{2^2\cdot 4^2\cdot 6^2 \cdots (2n)^2 \, (\cancel{\;})^2}$

$\therefore \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)^2}{(2n+1)^2}$ $u_{n+1} = \dfrac{1^2\cdot 3^2\cdot 5^2 \cdots (2n+1)^2}{2^2\cdot 4^2 \cdots (2n+2)^2}$

$$= \left[\frac{2n\left(1+\frac{1}{n}\right)}{2n\left(1+\frac{1}{2n}\right)}\right]^2 = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

i.e $\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^{2}\left(1+\frac{1}{2n}\right)^{-2}$

$= \left[1+\frac{2}{n}+\frac{1}{n^2}\right]\left[1-2\left(\frac{1}{2n}\right)+\frac{(-2)(-2-1)}{\lfloor 2}\left(\frac{1}{2n}\right)^2+\cdots\right]$

$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left[1-\frac{1}{n}+\frac{3}{4n^2}+\cdots\right]$

$= 1+\frac{1}{n}(2-1)+\frac{1}{n^2}\left[\frac{3}{4}-2+1\right]+\cdots$

$= 1+\frac{1}{n}+\frac{1}{n^2}\left[-\frac{1}{4}+o\left(\frac{1}{n}\right)\right] \cdots\cdots$ (1)

Which is of the form

$\alpha+\frac{\beta}{n}+\frac{\lambda_n}{n^m}$ where $n\to\infty$, a finite quantity

Again from (1)

we find λ_n is a finite for all n and $m>1$

Also $\alpha = 1 = \beta$

thus By Gauss Test, the series

is ~~convergent~~ divergent.

[If $\frac{u_n}{u_{n+1}} = \alpha+\frac{\beta}{n}+\frac{\lambda_n}{n^m}$ where α, β are constant, $\forall n$ infinite then $m>1$ and λ_n infinite then the series is convergent]

and divergent if $\alpha>1$ if $\alpha<1$. If $\alpha=1$ then the series ~~is~~ Σu_n is convergent if $\beta>1$ and divergent if $\beta\le 1$

Problem 2.

Test the convergence the series

$$1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

Solution :→ Given series is

$$1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

Leaving the 1st Term, the series is

$$\frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

then $u_n = \dfrac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2n^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}$

$\therefore u_{n+1} = \dfrac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2n^2 \cdot (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2 (2n+3)^2}$

$\therefore \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+3)^2}{(2n+2)^2}$

$$= \left[\frac{2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{n}\right)}\right]^2 = \left(1+\frac{3}{2n}\right)^2 \left(1+\frac{1}{n}\right)^{-2}$$

$$= \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left[1-\frac{2}{n}+\frac{(-2)(-2-1)}{2}\cdot\frac{1}{n^2}+\cdots\right]$$

$$= \left[1+\frac{3}{n}+\frac{9}{4n^2}\right]\left[1-\frac{2}{n}+\frac{3}{n^2}-\cdots\right]$$

[Expanding by using Binomial theorem]

i.e $$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n^2}\left(3 - 6 + \frac{9}{4}\right) + \cdots$$

$$= 1 + \frac{1}{n} - \frac{3}{4}\cdot\frac{1}{n^2} + o\left(\frac{1}{n}\right) \quad \cdots (1)$$

which is of the form

$$\alpha + \frac{\beta}{n} + \frac{\lambda_n}{n^m} \quad \text{when } n \to \infty$$

$\lambda_n = -3/4$ = a finite quantity

Again from (1) we finite λ_n is finite for all n and $m > 1$

Also $\alpha = 1$ $\beta = 1$

Thus from Gauss's Test the given series $\sum u_n$ is divergent.

[Also we can find

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left[1 + o\left(\frac{1}{n}\right)\right] = 1$$

and so Raabe's Test fails.]

<u>Problem 3.</u> Test for convergence of hypergeometric series

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\,\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$$

for all positive values of x; α, β, γ being all positive.

Solution :-

Given series is

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\,\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$$

Leaving First term, if Σu_n be the Given series then

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot2\cdot3\cdots n\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)}$$

$$\Rightarrow u_{n+1} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n)\,\beta(\beta+1)\cdots(\beta+n)}{1\cdot2\cdot3\cdots(n+1)\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x}$$

$$= \frac{n^2(1+1/n)(1+\frac{\gamma}{n})}{n^2(\alpha/n+1)(\beta/n+1)}\cdot\frac{1}{x}$$

$$= \frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)}\cdot\frac{1}{x} \quad \text{--- (1)}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

$\therefore$ From Ratio Test $\sum u_n$ is convergent

if $\frac{1}{x} > 1 \Rightarrow x < 1$

and divergent if $\frac{1}{x} < 1 \Rightarrow x > 1$

If $x = 1$ Ratio Test Fails

and hence From (1)

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\gamma/n\right)\left(1+\alpha/n\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left[1-\alpha/n+\frac{\alpha^2}{n^2}\cdots\right]\left[1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\cdots\right]$$

$$= \left[1+\frac{1+\gamma}{n}+\frac{\gamma}{n^2}\right]\left[1-\frac{\alpha+\beta}{n}+0\left(\frac{1}{n^2}\right)\right]$$

$$= 1+\frac{1+\gamma-\alpha-\beta}{n}+0\left(\frac{1}{n^2}\right)$$

Which is of the form

$$\alpha+\frac{\beta}{n}+\frac{\lambda_n}{n^m}$$ and so

Gauss's Test

we conclude that the Given Series

$\sum u_n$ is convergent if $1+\gamma-\alpha-\beta > 1$ i.e $\gamma > \alpha+\beta$

and Divergent if $1+\gamma-\alpha-\beta \leq 1$ i.e $\gamma \leq \alpha+$

i.e